

Isocategorical Groups

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1 Introduction and the Main Results

1. It is well known that the category $\text{Rep}(G)$ of finite dimensional complex representations of a finite group G , regarded as a symmetric tensor category, uniquely determines the group G up to isomorphism (see e.g. [DM, Theorem 3.2]).

On the other hand, it is known that the Grothendieck ring of $\text{Rep}(G)$ (regarded as a ring with a distinguished basis, formed by the simple modules) is not sufficient to determine G . For instance, the two nonisomorphic nonabelian groups of order 8 – the group of symmetries of the square and the quaternion group – have the same Grothendieck rings.

This raises the question whether there is an intermediate amount of information between these two extremes that still determines G .

Motivated by this question, we introduce the following notion of isocategorical finite groups.

Definition 1.1 *Two finite groups G_1, G_2 are called isocategorical if $\text{Rep}(G_1)$ is equivalent to $\text{Rep}(G_2)$ as a tensor category (without regard for the symmetric structure).*

The property of two groups to be isocategorical is much stronger than the property to have the same Grothendieck rings (with a basis). For example, it is known (and will also follow from the results below) that the two nonabelian groups of order 8 are not isocategorical. This raises the question whether isocategorical groups must be isomorphic.

Unfortunately (or fortunately, depending on the reader's taste), the answer to this question is negative. Namely, we prove the following result.

Theorem 1.2 *Let Y be a vector space of dimension ≥ 3 over a field of two elements. Let $V := Y \oplus Y^*$ be the corresponding symplectic vector space, and let $\text{Sp}(V)$ denote the group of symplectic linear transformations of V . Then there exists a group G_b which is isocategorical but not isomorphic to $G := \text{Sp}(V) \ltimes V$.*

The group G_b , constructed in Section 4, is in fact well known in group theory. Namely, G_b is an extension of $\text{Sp}(V)$ by V (using a cohomology class b), described by R. Griess in 1973 [Gr]; this extension can be defined over any field but is nontrivial only in characteristic 2.

The group G_b contains the *pseudosymplectic group* $\text{Ps}(V)$ defined by A. Weil in 1964 [W], and in particular the “Weil representation” of $\text{Ps}(V)$ (a projective representation on the space of complex functions of Y) extends to G_b (while by [LS] the group $G := \text{Sp}(V) \ltimes V$ and even $\text{Sp}(V)$, for $\dim(Y) \geq 4$, does not admit nontrivial projective representations of such a small dimension).

We will call the group G_b the *affine pseudosymplectic group*, since it is a “modification” of the *affine symplectic group* $\text{ASp}(V) := \text{Sp}(V) \ltimes V$, and contains the pseudosymplectic group $\text{Ps}(V)$. Correspondingly, we will denote G_b by $\text{APs}(V)$.

2. In spite of Theorem 1.2, all groups isocategorical to a given group G can be explicitly classified in group-theoretical terms. Their classification is described by the following theorem, which is our second result.

Let A be a normal abelian subgroup of G of order 2^{2m} , and set $K := G/A$. Let $R : A^\vee \rightarrow A$ be a G -invariant skew-symmetric isomorphism between A and its character group A^\vee (i.e. $(Rx, x) = 1 \in \mathbf{C}^*$ for all $x \in A^\vee$). We can regard R as a G -invariant nondegenerate skew-symmetric bilinear form on A^\vee with values in \mathbf{C}^* . We will denote this form by the same letter R : $(x, y) \mapsto R(x, y) := (Rx, y)$.

The form R defines a class in $H^2(A^\vee, \mathbf{C}^*)^K$. Namely, this class is represented by any 2-cocycle J on A^\vee with values in \mathbf{C}^* , such that $R(x, y) = J(x, y)/J(y, x)$ (this class does not depend on the choice of J since for any two choices J_1, J_2 the cocycle $J_1 J_2^{-1}$ is symmetric and hence is a coboundary, see Lemma 2.3 below). We denote this class by \bar{R} .

Let

$$\tau : H^2(A^\vee, \mathbf{C}^*)^K \rightarrow H^2(K, A) \quad (1)$$

be the homomorphism defined as follows. For $c \in H^2(A^\vee, \mathbf{C}^*)^K$, let J be a 2-cocycle representing c . Then for any $g \in K$, the 2-cocycle $J^g J^{-1}$ is a coboundary (see Lemma 2.3 below). Choose a cochain $z(g) : A^\vee \rightarrow \mathbf{C}^*$ such that $dz(g) = J^g J^{-1}$. Let

$$\tilde{b}(g, h) := \frac{z(gh)}{z(g)z(h)^g}. \quad (2)$$

It is easy to see that for any $g, h \in K$, the function $\tilde{b}(g, h) : A^\vee \rightarrow \mathbf{C}^*$ is a character, i.e. $\tilde{b}(g, h)$ belongs to A (see Lemma 3.7 below). Thus, \tilde{b} can be regarded as a 2-cocycle of K with coefficients in A (our definition of the coboundary operator for group cohomology is the standard one, see [B], p.59). So \tilde{b} represents a class b in $H^2(K, A)$. It is easy to show that this class depends only on c and not on the choices we made. So we define τ by $\tau(c) = b$.

Let $b := \tau(\bar{R})$. Let \tilde{b} be any cocycle representing b . For any $\gamma \in G$, let $\bar{\gamma}$ be the image of γ in K . Introduce a new multiplication law $*$ on G by

$$\gamma_1 * \gamma_2 := \tilde{b}(\bar{\gamma}_1, \bar{\gamma}_2) \gamma_1 \gamma_2. \quad (3)$$

It is easy to show that this multiplication law introduces a new group structure on G , which (up to an isomorphism) depends only on b and not on \tilde{b} . Let us call this group G_b .

Theorem 1.3 *The following hold:*

1. *The group G_b is isocategorical to G .*
2. *Any group isocategorical to G is obtained in this way.*

Let us say that a finite group G is *categorically rigid* if any group isocategorical to G is actually isomorphic to G .

Corollary 1.4 *If G is not categorically rigid then G admits a normal abelian subgroup A , of order 2^{2m} , $m \geq 1$, equipped with a skew-symmetric G -invariant isomorphism $R : A^\vee \rightarrow A$. In particular, all groups of orders $2k+1$, $2(2k+1)$, and all simple groups are categorically rigid.*

Remark 1.5 Note that Corollary 1.4 implies that the quaternion group is categorically rigid (all its normal subgroups A of order 4 are cyclic and hence do not admit a skew symmetric isomorphism $A^\vee \rightarrow A$). In particular, the two nonabelian groups of order 8 are not isocategorical.

The structure of the paper is as follows. In Section 2 we introduce some tools from Hopf algebra theory, which are needed to prove Theorems 1.2 and 1.3. In Section 3, we prove Theorem 1.3. In Section 4, we prove Theorem 1.2. In Section 5, we use the notion of isocategorical groups to discuss the question when two triangular Hopf algebras are isomorphic as Hopf algebras. In Section 6, we discuss the properties of the affine pseudosymplectic group $\text{APs}(V)$.

We note that Theorem 1.3 and the results of section 5 can be generalized without significant changes to the case when finite groups are replaced by affine proalgebraic groups. We do not discuss the details of this generalization in this paper.

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2 Twists

2.1 Twists and Gauge Transformations

Let H be a Hopf algebra. The multiplication, unit, comultiplication, counit and antipode in H will be denoted by $m, 1, \Delta, \varepsilon, S$ respectively. We recall Drinfeld's notion of a *twist* for H [D].

Definition 2.1 *A twist for H is an invertible element $J \in H \otimes H$ which satisfies*

$$(\Delta \otimes I)(J)(J \otimes 1) = (I \otimes \Delta)(J)(1 \otimes J) \text{ and } (\varepsilon \otimes I)(J) = (I \otimes \varepsilon)(J) = 1, \quad (4)$$

where I is the identity map of H .

Given a twist J for H , one can define a new Hopf algebra structure $(H^J, m, 1, \Delta^J, \varepsilon, S^J)$ on the algebra $(H, m, 1)$ as follows. The coproduct is determined by

$$\Delta^J(a) = J^{-1} \Delta(a) J \text{ for any } a \in H, \quad (5)$$

and the antipode is determined by

$$S^J(a) = Q^{-1} S(a) Q \text{ for any } a \in H, \quad (6)$$

where $Q := m \circ (S \otimes I)(J)$.

If H is quasitriangular with the universal R -matrix R then so is H^J with the universal R -matrix $R^J := J_{21}^{-1} R J$.

If J is a twist for H and x is an invertible element of H such that $\varepsilon(x) = 1$, then

$$J^x := \Delta(x) J(x^{-1} \otimes x^{-1}) \quad (7)$$

is also a twist for H . We will call the twists J and J^x *gauge equivalent*, and x a *gauge transformation*.

Remark 2.2 We note that in [EG2] the condition $\varepsilon(x) = 1$ was accidentally omitted and should be added.

The map $(H^J, R^J) \rightarrow (H^{J^x}, R^{J^x})$ determined by $a \mapsto xax^{-1}$ is an isomorphism of quasitriangular Hopf algebras. Thus, replacing a twist by a gauge equivalent one does not change the isomorphism class of H^J as a quasitriangular Hopf algebra.

2.2 Twists for Group Algebras of Abelian Groups

Twists for group algebras of finite abelian groups will be of particular interest in this paper. So let us give a well known description of such twists in terms of group 2-cocycles.

Let A be a finite *abelian* group, and $A^\vee := \text{Hom}(A, \mathbf{C}^*)$ be its character group.

Then twists for $H := \mathbf{C}[A]$ are in one to one correspondence with 2-cocycles \tilde{J} of A^\vee with coefficients in \mathbf{C}^* , such that $\tilde{J}(0, 0) = 1$.

Indeed, let J be a twist for H , and define $\tilde{J} : A^\vee \times A^\vee \rightarrow \mathbf{C}^*$ via $\tilde{J}(\chi, \psi) := (\chi \otimes \psi)(J)$. It is straightforward to verify that \tilde{J} is a 2-cocycle of A^\vee (see e.g. [M, Proposition 3]), and that $\tilde{J}(0, 0) = 1$.

Conversely, let $\tilde{J} : A^\vee \times A^\vee \rightarrow \mathbf{C}^*$ be a 2-cocycle of A^\vee with coefficients in \mathbf{C}^* . For $\chi \in A^\vee$, let $E_\chi := |A|^{-1} \sum_{a \in A} \chi(a) a$ be the associated idempotent of H . Then it is straightforward to verify that if $\tilde{J}(0, 0) = 1$ then $J := \sum_{\chi, \psi \in A^\vee} \tilde{J}(\chi, \psi) E_\chi \otimes E_\psi$ is a twist for H (see e.g. [M, Proposition 3]). Moreover, it is easy to check that the above two assignments are inverse to each other.

From now on we will abuse notation and denote twists for $\mathbf{C}[A]$ and corresponding 2-cocycles by the same letter.

We note that gauge equivalence of twists for H corresponds to homological equivalence of the associated 2-cocycles. This fact will be useful later.

In conclusion let us prove a lemma that is used many times throughout the paper.

Lemma 2.3 *A symmetric 2-cocycle J of a finite abelian group Γ with coefficients in \mathbf{C}^* is a coboundary.*

Proof: This is a simple special case of Corollary 3.3 of [EG2]. However, let us give an elementary direct proof.

Recall that 2-cocycles encode abelian extensions. A symmetric cocycle defines an extension that is itself an abelian group. Therefore, the lemma claims that any exact sequence of abelian groups

$$0 \rightarrow \mathbf{C}^* \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow 0$$

is split. Passing to the dual groups, we obtain the statement that any exact sequence of abelian groups

$$0 \rightarrow E \rightarrow \tilde{E} \rightarrow \mathbb{Z} \rightarrow 0,$$

with $|E| < \infty$, is split. But this statement easily follows from the classification of finitely generated abelian groups (i.e. from the fact that \tilde{E} decomposes into a direct sum of a free part and the torsion part). ■

3 Proof of Theorem 1.3

Let us first prove part 2 of the theorem. We begin by reformulating the property of two groups to be isocategorical in Hopf algebraic terms. This is accomplished by the following lemma.

Lemma 3.1 *Finite groups G_1, G_2 are isocategorical if and only if there exists a twist J for $\mathbf{C}[G_1]$ such that $\mathbf{C}[G_1]^J$ and $\mathbf{C}[G_2]$ are isomorphic as Hopf algebras (but not necessarily as triangular Hopf algebras).*

Proof: The "if" direction is clear, since twisting of a Hopf algebra does not change the tensor category of its representations (see e.g. [ES, Proposition 13.3]). To prove the "only if" direction, let $F : \text{Rep}(G_1) \rightarrow \text{Rep}(G_2)$ be a tensor equivalence, and let F_2 be the forgetful functor on $\text{Rep}(G_2)$. Then $F_2 \circ F$ is a tensor (maybe nonsymmetric) fiber functor on $\text{Rep}(G_1)$, so $\text{End}(F_2 \circ F)$ is the triangular Hopf algebra $\mathbf{C}[G_1]^J$, where J is some twist for $\mathbf{C}[G_1]$. On the other hand, $\text{End}(F_2) = \mathbf{C}[G_2]$. So the functor F establishes an isomorphism of Hopf algebras $f : \mathbf{C}[G_1]^J \rightarrow \mathbf{C}[G_2]$ (but not necessarily of triangular Hopf algebras, since F is not assumed to be symmetric). We refer the reader to [Ge] for more explanations. ■

So let us assume that G_1, G_2 are isocategorical, and that such a twist J has been fixed.

Corollary 3.2 *The Hopf algebra $\mathbf{C}[G_1]^J$ is cocommutative.*

Proof: Clear. ■

Let $H_J \subseteq \mathbf{C}[G_1]^J$ denote the span of (left or right) tensorands of the R-matrix $R^J := J_{21}^{-1}J$.

Proposition 3.3 *H_J is a Hopf subalgebra of $\mathbf{C}[G_1]^J$, isomorphic to a group algebra of an abelian group.*

Proof: By the definition, H_J is a minimal triangular Hopf algebra [R], i.e. it is generated by the components of its R-matrix. Therefore, H_J is isomorphic, via the R-matrix, to its dual with opposite coproduct. But H_J is cocommutative by Corollary 3.2, as it is a Hopf subalgebra of $\mathbf{C}[G_1]^J$. Therefore, H_J is commutative. We are done. ■

Proposition 3.4 *There exists a twist \hat{J} for $\mathbf{C}[G_1]$ such that $\mathbf{C}[G_1]^J$ is isomorphic to $\mathbf{C}[G_1]^{\hat{J}}$ as triangular Hopf algebras, and $\hat{J} \in H_J \otimes H_J$.*

Proof: By construction, the Drinfeld element u of the triangular Hopf algebra (H_J, R^J) is 1. Therefore, it is easy to show that there exists a twist $J' \in H_J \otimes H_J$ such that $R^J = R^{J'}$ (this is a very simple special case of [EG1, Theorem 2.1], which can be proved directly, without the use of Deligne's theorem). Now, we see that $\mathbf{C}[G_1]^{J(J')^{-1}}$ is a triangular Hopf algebra whose R -matrix is $1 \otimes 1$. Thus, $J(J')^{-1}$ is a symmetric twist, hence it is gauge equivalent to 1 (see Corollary 3.3 of [EG2]). Thus, there exists an isomorphism of triangular Hopf algebras $f : \mathbf{C}[G_1]^{J(J')^{-1}} \rightarrow \mathbf{C}[G_1]$. Therefore, $f : \mathbf{C}[G_1]^J \rightarrow \mathbf{C}[G_1]^{f(J')}$ is an isomorphism of triangular Hopf algebras. It is clear that the twist $\hat{J} := f(J')$ satisfies our requirements. ■

Thus, we can assume, without loss of generality, that $J \in H_J \otimes H_J$. This implies that $H_J = \mathbf{C}[A]$, where A is an abelian subgroup of G_1 , and $J \in \mathbf{C}[A] \otimes \mathbf{C}[A]$.

Proposition 3.5 *A is a normal subgroup of G_1 , and the action of the group $K := G_1/A$ on A by conjugation preserves R^J .*

Proof: By cocommutativity of $\mathbf{C}[G_1]^J$ we get $J^{-1}(g \otimes g)J = J_{21}^{-1}(g \otimes g)J_{21}$ for all $g \in G_1$. This implies that R^J commutes with $g \otimes g$ (here we use that A is abelian and hence $R^J = JJ_{21}^{-1}$). Since the left (and right) tensorands of R^J span $\mathbf{C}[A]$, the result follows. ■

Lemma 3.6 *The twist J can be chosen in such a way that $|A| = 2^{2m}$, $m \geq 0$.*

Proof: Write $A = A_0 \times A_1$, where A_0 is the 2-Sylow subgroup of A , and the subgroup A_1 has odd order. Recall that twists for $\mathbf{C}[A]$ are 2-cocycles of A^\vee with coefficients in \mathbf{C}^* , and that gauge equivalence of twists corresponds to homological equivalence of 2-cocycles. It is well known that $H^2(A, \mathbf{C}^*) = H^2(A_0, \mathbf{C}^*) \oplus H^2(A_1, \mathbf{C}^*)$ (since the orders of A_0, A_1 are relatively prime, see e.g. [CE]), so we can assume that $J = J_0 J_1$, where $J_0 \in \mathbf{C}[A_0]^{\otimes 2}$, and $J_1 \in \mathbf{C}[A_1]^{\otimes 2}$. Consequently, the R-matrix $R := R^J$ has a factorization $R = R_0 R_1$, where $R_0 := R^{J_0}$, and $R_1 := R^{J_1}$.

For all $x, y \in A_1$, define $J'_1(x, y) := R_1(x/2, y)$ (this is well defined since A_1 has odd order). Let R'_1 be the R -matrix associated with the twist J'_1 . Then

$$\begin{aligned} R'_1(x, y) &= R_1(y/2, x)^{-1} R_1(x/2, y) = R_1(x, y/2) R_1(x/2, y) \\ &= R_1(x/2, y/2)^2 R_1(x/2, y) = R_1(x/2, y)^2 = R_1(x, y). \end{aligned}$$

This implies that the R -matrix associated with the twist $(J'_1)^{-1} J_1$ is equal to $1 \otimes 1$. Hence, this twist is symmetric, and therefore is a coboundary (i.e. gauge equivalent to the identity), by Lemma 2.3. We thus conclude that the twist $(J'_1)^{-1} J$ is gauge equivalent to J_0 .

However, it is clear that J'_1 is G_1 -invariant (since R_1 is), hence $\mathbf{C}[G_1]^J$ and $\mathbf{C}[G_1]^{(J'_1)^{-1}J}$ are isomorphic as Hopf algebras, and the result follows. ■

Thus, we will further assume that $|A| = 2^{2m}$, $m \geq 0$.

Now we are ready to present the concluding part of the proof of part 2 of Theorem 1.3. This part will consist in calculating the group of grouplike elements of $\mathbf{C}[G_1]^J$, and showing that it is isomorphic to $(G_1)_b$ for an appropriate cohomology class b .

We will view J not only as a twist but also as a 2-cocycle of A^\vee with values in \mathbf{C}^* , according to Section 2. For $g \in K$ let us write J^g for the action of g on J . Since R^J is invariant under G_1 , we get

$$J^g J^{-1} = J_{21}^g J_{21}^{-1}.$$

This implies that the 2-cocycle $J^g J^{-1}$ of A^\vee is symmetric, hence it is a coboundary, by Lemma 2.3. Thus, there exists a function $z : K \rightarrow \mathbf{C}[A]^\times$, $g \mapsto z(g)$, (where $\mathbf{C}[A]^\times$ is the group of invertible elements of $\mathbf{C}[A]$), such that

$$J^g J^{-1} = \Delta(z(g))(z(g)^{-1} \otimes z(g)^{-1}).$$

For $\gamma \in G_1$, let $\bar{\gamma}$ denote its image in K . Let us write $z(\gamma)$ for $z(\bar{\gamma})$. We have

$$\Delta^J(\gamma) = J^{-1} J^{\bar{\gamma}}(\gamma \otimes \gamma) = \Delta(z(\gamma))(z(\gamma)^{-1} \gamma \otimes z(\gamma)^{-1} \gamma).$$

This implies that $z(\gamma)^{-1} \gamma$ is a grouplike element in $\mathbf{C}[G_1]^J$. Thus, identifying $\mathbf{C}[G_1]^J$ with $\mathbf{C}[G_2]$, we obtain a map (of sets) $\varphi : G_1 \rightarrow G_2$, $\varphi(\gamma) = z(\gamma)^{-1} \gamma$.

It is easy to see that φ is injective (and hence bijective). Indeed, $\varphi(\gamma_1) = \varphi(\gamma_2)$ if and only if $z(\gamma_1)z(\gamma_2)^{-1} = \gamma_1\gamma_2^{-1}$. Since $z(\gamma_1)z(\gamma_2)^{-1} \in \mathbf{C}[A]$ and $\gamma_1\gamma_2^{-1} \in G_1$ we have that $\gamma_1 = a\gamma_2$ for some $a \in A$. But then, $z(\gamma_1) = z(a\gamma_2) = z(\gamma_2)$, hence $\gamma_1\gamma_2^{-1} = 1$.

Finally, it is obvious from the definition of φ that

$$\varphi(\gamma_1)\varphi(\gamma_2) = \tilde{b}(\bar{\gamma}_1, \bar{\gamma}_2)\varphi(\gamma_1\gamma_2),$$

where $\tilde{b}(g, h) = z(gh)/z(g)z(h)^g \in \mathbf{C}[A]^\times$.

Lemma 3.7 *The element $\tilde{b} := \tilde{b}(g, h)$ is a grouplike element, i.e. it belongs to A .*

Proof:

$$\Delta(\tilde{b})(\tilde{b}^{-1} \otimes \tilde{b}^{-1}) = \frac{J^{gh} J^{-1}}{J^g J^{-1} \cdot J^{gh} (J^g)^{-1}} = 1.$$

■

It is clear that \tilde{b} is a 2-cocycle of K with coefficients in A . Let b be the cohomology class of \tilde{b} in $H^2(K, A)$. We have shown that

$$\varphi(\gamma_1)\varphi(\gamma_2) = \varphi(\gamma_1 * \gamma_2),$$

i.e. that φ is an isomorphism $(G_1)_b \rightarrow G_2$. This completes the proof of part 2 of Theorem 1.3, since by the definition of b we have $b = \tau(\bar{R}^J)$.

Now let us prove part 1 of Theorem 1.3. This part is essentially obvious from what we have done. Namely, if G is a finite group, A its normal abelian subgroup, $K := G/A$, and $b \in H^2(K, A)$ is given by $b = \tau(\bar{R})$, then choose a twist $J \in \mathbf{C}[A]^{\otimes 2}$ such that $R = J_{21}^{-1} J$. Then, as we have shown above, $\mathbf{C}[G]^J$ is isomorphic as a Hopf algebra to $\mathbf{C}[G_b]$, and hence by Lemma 3.1, the groups G and G_b are isocategorical. Theorem 1.3 is proved.

Remark 3.8 It is clear that in Theorem 1.3, the ground field \mathbf{C} (over which the tensor categories are considered) can be replaced by any algebraically closed field of characteristic zero. On the other hand, one may introduce the following definition.

Definition 3.9 *Two finite groups G_1, G_2 are p -isocategorical (for a prime p) if their tensor categories of representations over an algebraically closed field of characteristic p are equivalent.*

Then we have the following.

Theorem 3.10 *Let G_1, G_2 be finite groups of order n prime to p . Then G_1, G_2 are isocategorical if and only if they are p -isocategorical. In other words, the criterion for G_1, G_2 to be p -isocategorical is the same as in Theorem 1.3.*

The theorem is proved in the same way as Theorem 1.3.

4 Proof of Theorem 1.2

Let F be the field of two elements. Let Y be an n -dimensional vector space over F , and $V := Y \oplus Y^*$ be the corresponding symplectic space. Let $G := \mathrm{Sp}(V) \ltimes V$. Let $A := V$, and $K := G/A = \mathrm{Sp}(V)$. Let $R : A^\vee \rightarrow A$ be the skew-symmetric isomorphism defined by the symplectic form. Let $b := \tau(\bar{R}) \in H^2(K, A)$.

The key step in the proof of Theorem 1.2 is the following result.

Theorem 4.1 *The element b is nonzero if $n \geq 3$.*

Before proving Theorem 4.1, let us show why it implies Theorem 1.2. For this, it is enough to show that if $b \neq 0$ then G_b is not isomorphic to G as a group.

First of all, note that K is a simple group (see [Go]). This implies that A is the unique normal subgroup of G of order $|A|$ (indeed, if there is another one, it will project to a non-trivial normal subgroup of K , which is impossible). Thus, if $\psi : G \rightarrow G_b$ is an isomorphism, then $\psi(A) = A$. But this implies that ψ is not only an isomorphism of groups but also an isomorphism of extensions of K by A , which means that b must be zero, as desired.

The rest of the section is devoted to the proof of Theorem 4.1.

Let P be the subgroup of K which preserves the subspace $Y^* \subset V$. It is sufficient to show that the restriction of b to P is nonzero. We will denote this restriction by b_P (i.e. $b_P \in H^2(P, V)$).

The group P is naturally identified with the semidirect product $L \ltimes U$, where $L := \mathrm{GL}(Y)$ and $U := S^2 Y^*$ is the additive group of all self adjoint operators $Y \rightarrow Y^*$. The actions of L and U on V are given by $l(v, f) = (lv, (l^*)^{-1}f)$, $l \in L$, and $u(v, f) = (v, f + uv)$, $u \in U$ (here $v \in Y$, $f \in Y^*$).

Let us introduce a basis of Y : e_1, \dots, e_n . An element $v \in Y$ can be written as $v = \sum y_j e_j$, where $y_j = y_j(v)$ are the coordinates of v .

Define a function $\tilde{\beta} : P \times P \rightarrow Y^*$ by the formula

$$\tilde{\beta}(u_1 l_1, u_2 l_2) = \chi(u_1, l_1 u_2 l_1^{-1}), \quad (8)$$

where $\chi : U \times U \rightarrow Y^*$ is defined by the formula

$$\chi(u, u')(e_j) = (-1)^{\frac{1}{2}[(u+u')_{jj} - u_{jj} - u'_{jj}]}, \quad (9)$$

in which $u_{ij} \in \{0, 1\}$ are the matrix elements of u , and the terms on the right hand side (e.g. u_{jj}) are understood as **integers**, 0 or 1 (i.e. not as elements of F).

Lemma 4.2 *The following hold:*

1. The element $\tilde{\beta}$ is a 2-cocycle of P with coefficients in Y^* , representing some cohomology class $\beta \in H^2(P, Y^*)$.
2. The cohomology class b_P is the image of β under the map of cohomology induced by the embedding of P -modules $Y^* \hookrightarrow V$.

Proof: Let us find a 2-cocycle representing b_P . To do this, we pick J to be $J((v, f), (v', f')) = (-1)^{\langle v, f' \rangle}$. Then $J^{ul} J^{-1}((v, f), (v', f')) = (-1)^{\langle v, uv' \rangle}$ for all $u \in U$ and $l \in L$.

Thus, we can take the function z to be

$$z(ul)(v, f) = (-1)^{\sum_{m < j} u_{mj} y_m y_j} \cdot i^{\sum_j u_{jj} y_j^2},$$

where $i := \sqrt{-1}$ (the exponent of i is regarded as an integer, 0 or 1). Therefore, we see that the function

$$\tilde{b}(u_1 l_1, u_2 l_2) = \frac{z(u_1 l_1 u_2 l_2)}{z(u_1 l_1) z(u_2 l_2)^{u_1 l_1}}$$

is given by

$$\tilde{b}(u_1 l_1, u_2 l_2) = \chi(u_1, l_1 u_2 l_1^{-1}),$$

where

$$\chi(u, u')(e_j) = i^{(u+u')_{jj} - u_{jj} - u'_{jj}},$$

where the terms on the right hand side are understood as integers. This concludes the proof of the lemma. \blacksquare

Now recall that we have a short exact sequence of P -modules

$$0 \rightarrow Y^* \rightarrow V \rightarrow Y \rightarrow 0$$

(where U acts trivially on both Y and Y^*). To this sequence there corresponds a long exact sequence of cohomology. In particular, this long exact sequence has a portion

$$H^1(P, Y) \rightarrow H^2(P, Y^*) \rightarrow H^2(P, V). \quad (10)$$

Thus, by Lemma 4.2, in order to show that $b_P \neq 0$ (and hence to prove Theorem 1.2), it is sufficient to prove the following proposition.

Proposition 4.3 *The element β is **not** contained in the image of $H^1(P, Y)$ under the connecting homomorphism $H^1(P, Y) \rightarrow H^2(P, Y^*)$ of the long exact sequence.*

The rest of this section is devoted to the proof of this proposition.

Let $\beta_U \in H^2(U, Y^*)$ be the restriction of β to U .

Lemma 4.4 $\beta_U \neq 0$.

Proof: It is easy to see from the explicit formula for $\tilde{\beta}$, given in (8), that β_U is nonzero after restriction to the 1-dimensional subspace of U generated by the element $u^{(11)}$, given by $u_{mj}^{(11)} := \delta_{1m}\delta_{1j}$ for all m, j . ■

Now we are ready to continue the proof of Proposition 4.3. Assume the contrary, i.e. that β is the image of $\alpha \in H^1(P, Y)$. Let $\alpha_U \in H^1(U, Y)$ be the restriction of α to U . Then β_U is the image of α_U under the connecting homomorphism $H^1(U, Y) \rightarrow H^2(U, Y^*)$. Thus, to derive a contradiction, it is sufficient to show that $\alpha_U = 0$.

It is clear that the element α_U is L -invariant, since it is obtained by restricting a cohomology class of $P = L \ltimes U$ (this follows from the fact that the action of any group on its cochain complex with any coefficients descends to the trivial action on cohomology). Therefore, the statement that $\alpha_U = 0$, and hence Proposition 4.3, follows from

Lemma 4.5 *If $n \geq 3$ then $H^1(U, Y)^L = 0$.*

Proof: Since U acts trivially on Y , we have $H^1(U, Y) = \text{Hom}_F(U, Y)$. Since F has two elements, the space $U = S^2Y^*$ of symmetric bilinear forms on Y is not an irreducible representation of $L = \text{GL}(Y)$. Namely, it has a submodule Λ^2Y^* (skew-symmetric bilinear forms on Y , i.e. forms B such that $B(v, v) = 0$ for all $v \in Y$), and the quotient is Y^* (the map $S^2Y^* \rightarrow Y^*$ is given by $B \mapsto B'$, where $B'(v) := B(v, v)$ for all $v \in Y$). It is easy to check (using explicit calculations with matrices) that the representations Y^* , Λ^2Y^* of $\text{GL}(Y)$ are irreducible, and that S^2Y^* is a nontrivial extension of one by the other. Therefore, an L -invariant homomorphism $U \rightarrow Y$ would have to be the pullback of an isomorphism $Y^* \rightarrow Y$. It is easy to show that such an isomorphism of L -modules exists only for $n = 1, 2$. Thus, for $n \geq 3$ we have $H^1(U, Y)^L = 0$, as desired. ■

Proposition 4.3, Theorem 4.1, and hence Theorem 1.2 are proved.

As we already mentioned in the introduction, we will call the group G_b the *affine pseudosymplectic group*, and denote it by $\text{APs}(V)$. The properties of this group are discussed in Section 6.

Remark 4.6 It is easy to see that the construction of the group $\text{APs}(V)$ makes sense over any finite field. However, it is easy to show that if the characteristic of the ground field is odd then $\text{APs}(V)$ is isomorphic to $\text{ASp}(V) := \text{Sp}(V) \ltimes V$.

Remark 4.7 The cohomology class β_U , which appears in the proof of Theorem 1.2, is invariant under $L = \text{GL}(Y)$. However, our construction of this class used a basis of Y . So let us give a basis-free construction of this class.

Recall that if U is a vector space over F , then $H^2(U, F) = (S^2U)^*$. Thus, $H^2(U, Y^*) = (S^2U)^* \otimes Y^* = \text{Hom}_F(S^2U, Y^*)$. Since we have an invariant projection $U \rightarrow Y^*$, we have an invariant projection $S^2U \rightarrow S^2Y^*$. Composing this projection with the invariant projection $S^2Y^* \rightarrow Y^*$, we get a $\text{GL}(Y)$ invariant map $S^2U \rightarrow Y^*$. One can show by a direct calculation that this is exactly the cocycle β_U .

5 Hopf Algebra Isomorphisms of Triangular Semisimple Hopf Algebras

Recall that in [EG2], we classified triangular semisimple Hopf algebras over \mathbf{C} , up to a triangular Hopf algebra isomorphism. In this section we will discuss a natural question which was not discussed in [EG2]: when are two triangular semisimple Hopf algebras over \mathbf{C} isomorphic as usual Hopf algebras (i.e. without regard for the triangular structure)? It turns out that the notion of isocategorical groups is useful in deciding this question.

Let B_1, B_2 be two triangular semisimple Hopf algebras over \mathbf{C} . According to [EG1], $B_1 = \mathbf{C}[G_1]^{J_1}$ and $B_2 = \mathbf{C}[G_2]^{J_2}$ as Hopf algebras, where G_i is a finite group, and J_i is a twist for $\mathbf{C}[G_i]$, $i = 1, 2$. Thus, in this section we will study the question: when are $\mathbf{C}[G_1]^{J_1}$, $\mathbf{C}[G_2]^{J_2}$ isomorphic as Hopf algebras?

Recall that it follows from [EG1, EG2] that $\mathbf{C}[G_1]^{J_1}$, $\mathbf{C}[G_2]^{J_2}$ are isomorphic as **triangular** Hopf algebras if and only if there exists a group isomorphism $f : G_1 \rightarrow G_2$ such that $f^{-1}(J_2)$ is gauge equivalent to J_1 . To formulate a similar criterion for the rougher equivalence relation of isomorphism as usual Hopf algebras, we need to generalize the notion of isomorphism of groups, and introduce the notion of a categorical isomorphism.

Definition 5.1 *Let G_1, G_2 be finite groups. A categorical isomorphism $G_1 \rightarrow G_2$ is a triple (J, A, f) , where A is an abelian normal subgroup of G_1 , $J \in \mathbf{C}[A]^{\otimes 2}$ a twist such that $R^J := J_{21}^{-1}J$ is G_1 -invariant and of maximal rank, and $f : \mathbf{C}[G_1]^J \rightarrow \mathbf{C}[G_2]$ an isomorphism of Hopf algebras.*

It is clear that two finite groups are isocategorical if and only if there is a categorical isomorphism between them.

There is an obvious notion of gauge equivalence of categorical isomorphisms, and for any G_1, G_2 there are finitely many categorical isomorphisms up to a gauge equivalence.

An example of a categorical isomorphism is a triple $(1, 1, f)$, where f is induced by an ordinary group isomorphism. Any categorical isomorphism which is gauge equivalent to such will be called *trivial*. It is clear that for a trivial categorical isomorphism $(1, J, f)$, the map f is actually an isomorphism of **triangular** Hopf algebras.

Theorem 5.2 *Any isomorphism of Hopf algebras $\phi : \mathbf{C}[G_1]^{J_1} \rightarrow \mathbf{C}[G_2]^{J_2}$ is representable in the form $f \circ \psi$, where $(J, A, f) : G_1 \rightarrow G_2$ is a categorical isomorphism, and $\psi : \mathbf{C}[G_1]^{J_1} \rightarrow \mathbf{C}[G_1]^{Jf^{-1}(J_2)}$ is an isomorphism of triangular Hopf algebras. In particular, $\mathbf{C}[G_1]^{J_1}$, $\mathbf{C}[G_2]^{J_2}$ are isomorphic as Hopf algebras if and only if there exists a categorical isomorphism $(J, A, f) : G_1 \rightarrow G_2$, such that the twist $Jf^{-1}(J_2)$ is gauge equivalent to J_1 .*

Proof: The map ϕ defines a Hopf algebra isomorphism $\mathbf{C}[G_1]^{J_1\phi^{-1}(J_2)^{-1}} \rightarrow \mathbf{C}[G_2]$. By the results of Section 3, there exists a categorical isomorphism $(J, A, f) : G_1 \rightarrow G_2$, and a

triangular Hopf algebra isomorphism $\psi : \mathbf{C}[G_1]^{J_1\phi^{-1}(J_2)^{-1}} \rightarrow \mathbf{C}[G_1]^J$ such that $\phi = f \circ \psi$. The theorem is proved. ■

Let us say that a finite group G is *strongly categorically rigid* if any its normal abelian subgroup A , which possesses a nondegenerate G -invariant skew-symmetric isomorphism $R : A^\vee \rightarrow A$, is trivial. For example, any simple group is strongly categorically rigid.

Corollary 5.3 *Suppose that G_1 or G_2 is strongly categorically rigid. Then any isomorphism of Hopf algebras $\mathbf{C}[G_1]^{J_1} \rightarrow \mathbf{C}[G_2]^{J_2}$ is actually an isomorphism of triangular Hopf algebras.*

Proof: Assume that G_1 is strongly categorically rigid. Then any categorical isomorphism $(J, A, f) : G_1 \rightarrow G_2$ is trivial. Therefore f is a triangular Hopf algebra isomorphism, and the result follows. ■

6 The Affine Pseudosymplectic Group

The goal of this section is to discuss the properties of the group $\text{APs}(V)$, constructed in Section 4. These results are well known and classical in the theory of finite groups; so the nature of this section is largely pedagogical. We note that much of what is discussed below was explained to us by R. Guralnick.

1. We start, following Weil [W], by constructing a projective representation of $\text{ASp}(V)$ called the “Weil representation”. We will take the ground field F to be the field of 2 elements, but the construction makes sense over any finite field, yielding a projective representation of $\text{Sp}(V) \ltimes V$ in odd characteristic.

Let $\mathcal{H} := \text{Fun}(Y, \mathbf{C})$ be the space of complex valued functions on Y . Consider the projective action ρ of V on \mathcal{H} given by

$$(\rho(y)f)(x) := f(x+y), \quad (\rho(y^*)f)(x) := (-1)^{y^*(x)}f(x), \quad y \in Y, \quad y^* \in Y^*, \quad \rho(y, y^*) := \rho(y^*)\rho(y).$$

Then

$$\rho(y_1, y_1^*)\rho(y_2, y_2^*) = \rho(y_1 + y_2, y_1^* + y_2^*)(-1)^{y_2^*(y_1)}.$$

This shows that,

$$\rho(v_1)\rho(v_2) = \rho(v_1 + v_2)J(v_1, v_2),$$

where J is the 2-cocycle defined in the proof of Lemma 4.2. In other words, ρ extends to a representation of the “Heisenberg” group E , which consists of pairs (v, c) , $v \in V, c \in \{\pm 1\}$, with multiplication law $(v_1, c_1) \cdot (v_2, c_2) = (v_1 + v_2, c_1c_2J(v_1, v_2))$.

Let $g \in \text{Sp}(V)$. Consider the assignment ρ^g defined by $\rho^g(v) := \rho(g^{-1}v)$. Then

$$\rho^g(v_1)\rho^g(v_2) = \rho^g(v_1 + v_2)J^g(v_1, v_2).$$

Let $z(g) : V \rightarrow \mathbf{C}^*$ be as in the proof of Lemma 4.2. Consider the map $\tilde{\rho}^g(v) := \rho^g(v)z(g)(v)$. Then $\tilde{\rho}^g(v)$ also extends to a representation of E .

It is easy to see that the representations $\rho, \tilde{\rho}^g$ of E are isomorphic (both are irreducible of dimension $2^{\dim(Y)}$, and such a representation is unique). Therefore, there exists a unique, up to scaling, isomorphism $A(g) : \mathcal{H} \rightarrow \mathcal{H}$, such that $A(g)\tilde{\rho}^g(v) = \rho(v)A(g)$.

For a character $\chi : V \rightarrow \mathbf{C}^*$, let T_χ be an isomorphism $\mathcal{H} \rightarrow \mathcal{H}$, defined by

$$\rho(w)T_\chi = T_\chi\rho(w)\chi(w), \quad w \in V.$$

These operators exist and are unique up to scaling for the same reasons as $A(g)$.

Proposition 6.1 *One has*

$$T_\chi T_\xi = c_1(\chi, \xi)T_{\chi+\xi}, \quad A(g)T_\chi = c_2(g, \chi)T_{g\chi}A(g),$$

$$A(gh) = c_3(g, h)T_{\tilde{b}(g,h)}A(g)A(h),$$

where \tilde{b} is given in (2), and c_1, c_2, c_3 are suitable nonvanishing complex functions (depending on the choice of the normalization of $A(g), T_\chi$).

Proof: This follows from Schur's lemma, since the ratios of the RHS and the LHS commute with $\rho(v)$. ■

This shows that the assignment $A(\chi, g) := T_\chi A(g)$ defines a projective representation of the group $\text{APs}(V)$ on \mathcal{H} , which may be called the “Weil representation”, by analogy with [W].

Remark 6.2 The group $\text{Hom}(V, \mathbf{C}^*)$ is naturally identified with V via $v \mapsto \chi_v$, $v \in V$, where $\chi_v(w) = (-1)^{\langle v, w \rangle}$.

2. Here is another proof of Theorem 1.2 for $\dim(Y) \geq 4$. It was shown in [LS] that for $\dim(Y) \geq 4$, the group $\text{Sp}(V)$ (and hence $\text{ASp}(V) = \text{Sp}(V) \ltimes V$) has no irreducible nontrivial projective representations of degree $2^{\dim(Y)}$ (or smaller). Thus, $\text{APs}(V)$ is not isomorphic to $\text{ASp}(V)$ (as it has the Weil representation).

In particular, this argument shows that the statement that $\text{ASp}(V)$ and $\text{APs}(V)$ have equivalent representation categories, does NOT extend to projective representations.

3. Here is another construction of the group $\text{APs}(V)$, given in [Gr].

Let T be the group $(E \times \mathbb{Z}/4\mathbb{Z})/\Gamma$, where Γ is the subgroup of order 2 generated by the element $(-1, 2)$, $-1 \in E$, $2 \in \mathbb{Z}/4\mathbb{Z}$. Let $\text{Aut}_0(T)$ be the group of all automorphisms of T that act trivially on $\mathbb{Z}/4\mathbb{Z}$. Then $\text{Aut}_0(T)$ contains the subgroup $\text{Inn}(T) = T/Z(T)$ of inner automorphisms of T , which is isomorphic to V . It is easy to check (see [Gr]) that the group

$\text{Out}_0(T) = \text{Aut}_0(T)/\text{Inn}(T)$ is isomorphic to $\text{Sp}(V)$. Thus, $\text{Aut}_0(T)$ is an extension of $\text{Sp}(V)$ by V .

The group T has a unique irreducible representation \mathcal{H} of dimension $2^{\dim(Y)}$ with a fixed action of $Z(T) = \mathbb{Z}/4\mathbb{Z}$ (i.e. the generator acting by multiplication by i). Therefore, the group $\text{Aut}_0(T)$ acts projectively in \mathcal{H} .

Proposition 6.3 *The group $\text{Aut}_0(T)$ is isomorphic to $\text{APs}(V)$, and the projective representation \mathcal{H} of $\text{Aut}_0(T)$ is the Weil representation.*

Proof: The proof is obtained directly from comparing the two constructions of the Weil representation. ■

With this construction of $\text{APs}(V)$, the essential part of the proof of Theorem 1.2 (i.e. the nontriviality of the extension) was proved in [Gr] (Corollary 2).

4. Recall [W] that the pseudosymplectic group $\text{Ps}(V)$ is the group of all pairs (g, Q) , where Q is a quadratic form on V , and g is a linear transformation of V , such that

$$Q(x + y) - Q(x) - Q(y) = J(gx, gy) - J(x, y),$$

with the operation $(g, Q)(g', Q') = (Q^{(g')^{-1}} + Q', gg')$. For fields of odd characteristic, this group is isomorphic to $\text{Sp}(V)$.

Identify T with $V \times \mathbb{Z}/4\mathbb{Z}$ in a natural way. The group $\text{Ps}(V)$ acts on T in the following way:

$$(g, Q)(v, z) = (gv, z + 2Q(v)).$$

Thus, $\text{Ps}(V)$ is a subgroup of $\text{APs}(V)$. Namely, one can show that it is the preimage, under the projection $\text{APs}(V) \rightarrow \text{Sp}(V)$, of the orthogonal group $\text{O}(V)$ of the quadratic form $J(x, x)$. In particular, $\text{Ps}(V)$ has the Weil representation, constructed in [W] for local fields, and known even earlier for finite fields (see e.g. [BRW]).

Remark 6.4 In fact, as shown in [Gr],[BRW], the group $\text{Ps}(V)$, for large enough $\dim(Y)$, is also a nontrivial extension (of $\text{O}(V)$ by V).

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